Lecture 19

16.6 - Parametric Surfaces

Suppose we have a surface S in $\mathbb{R}^3$. Since a surface is inherently 2-dimensional, it requires 2 variables to parametrize. A parametrization of S looks likes $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$, $(u,v) \in D$. where D is the domain.

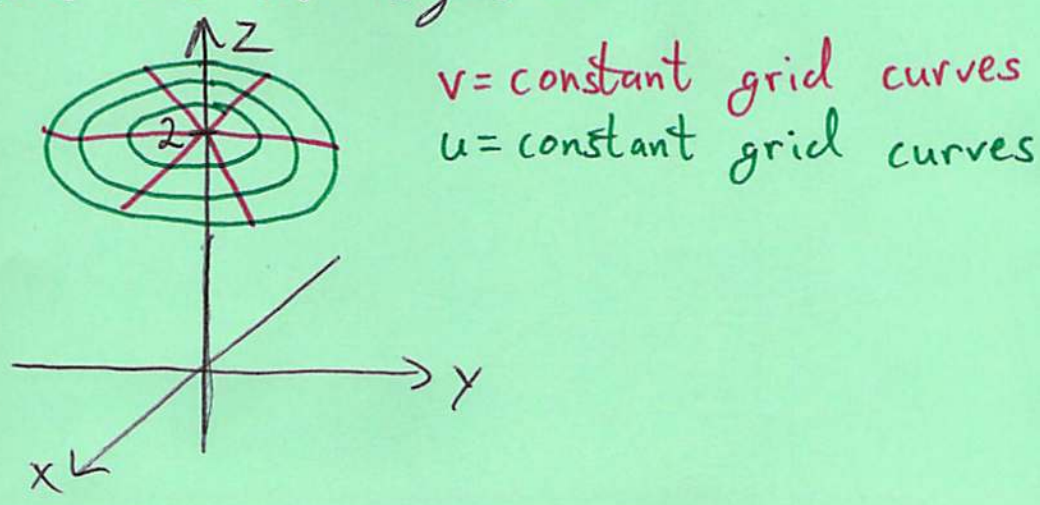
Ex: Identify and sketch the surface parametrized by:

a) $\vec{r}(u,v) = \langle u\cos v, u\sin v, 2 \rangle$, $0 \le u \le 1$, $0 \le v \le 2\pi$

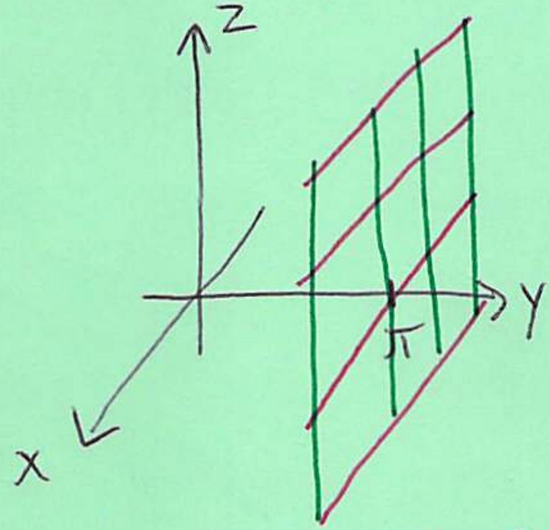
b) $\vec{r}(s,t) = \langle s, \pi, t \rangle$, $s,t \in \mathbb{R}$

c) $\vec{r}(u,v) = \langle u, v, u^2+v^2 \rangle$, $u,v \in \mathbb{R}$

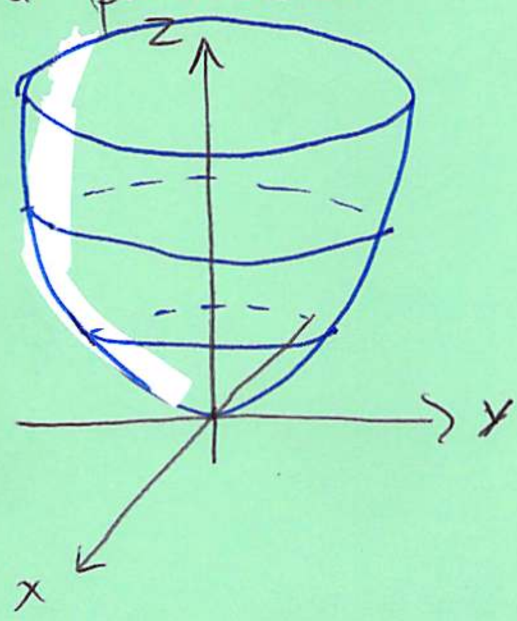
Sol: a) Notice $x^2+y^2 = u^2\cos^2 v + u^2 \sin^2 v = u^2$. So, since u goes from 0 to 1, we get a whole disk, and $z=2$ means it's at height 2.



b) Here x & z can be anything, while y is stuck at π. So, the graph is the plane $y=\pi$:



c) The components satisfy $x^2+y^2=z$, so the surface is a paraboloid:



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Recall that the paraboloid is described in cylindrical by $z=r^2$, so a reparametrization of (c) is:

$$\vec{r}(r,\theta)=\langle r\cos\theta, r\sin\theta, r^2\rangle,\quad r\geq 0,\ 0\leq\theta\leq 2\pi$$

Suppose we have a surface given by $z=f(x,y)$, $(x,y)\in D$.
Since z is completely determined by x&y, we can let x&y be the parameters, then:

$$\vec{r}(x,y)=\langle x,y,f(x,y)\rangle,\ (x,y)\in D.$$

is a parametrization of the surface.

Ex: Parametrize the surface $z=3\sqrt{x^2+y^2}$.

Sol: Using the above, this is parametrized by

$$\vec{r}(x,y)=\langle x,y,3\sqrt{x^2+y^2}\rangle.$$

However, to give it nicer grid curves, it's better to use cylindrical coordinates: So $z=3r$ and:

$$\vec{r}(r,\theta)=\langle r\cos\theta, r\sin\theta, 3r\rangle,\ 0\le r,\ 0\le\theta\le 2\pi.$$

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Ex: Parametrize the sphere $x^2+y^2+z^2=81$

Sol: Since we cannot write down the sphere as the graph of a function, we have to try something else. Luckily, we have a coordinate system which nicely describes spheres: spherical coordinates. This sphere is given by $\rho=9$. Using the equations from spherical coordinates we find <u>parametric equations</u> for the sphere

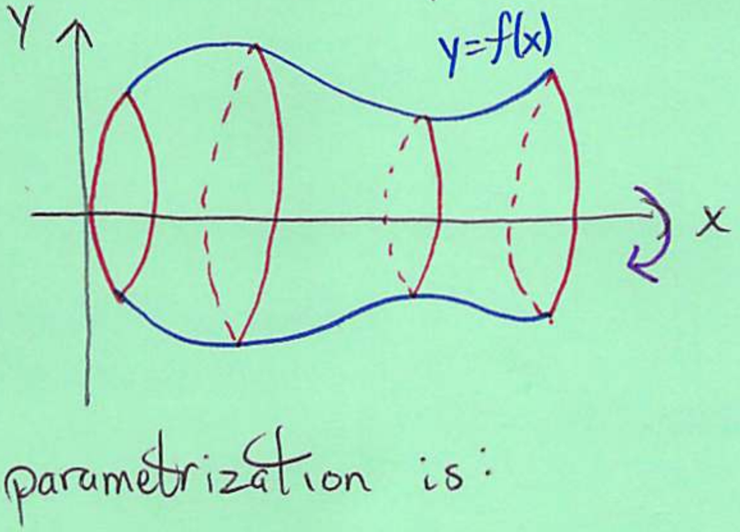
$$x = 9\cos\theta\sin\varphi$$
$$y = 9\sin\theta\sin\varphi \;,\quad 0 \le \theta \le 2\pi,\; 0 \le \varphi \le \pi.$$
$$z = 9\cos\varphi$$

$\Diamond$

Another example of a parametrization which revisits chapter 12 is that of a plane. A plane passing through a point P w/ vectors $\vec{a}$ & $\vec{b}$ (nonzero, nonparallel) in the plane is parametrized by:

$$\vec{r}(s,t) = \overrightarrow{OP} + s\vec{a} + t\vec{b}\,.$$

One more example is <u>surfaces of revolution</u>. To obtain this, we rotate a graph $y = f(x)$ about the x-axis, for example.



A parametrization is:

$$\vec{r}(x,\theta) = \langle x, f(x)\cos\theta, f(x)\sin\theta \rangle \;,\quad 0 \le \theta \le 2\pi$$

Let's revisit another earlier problem: that of finding tangent planes to surfaces. Recall that to find tangent vectors, we needed curves in the surface, passing through the desired point, then took derivatives. If the surface is parametrized by

$$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$$

and the point is $P_0 = \vec{r}(u_0, v_0)$, then we can take the two grid curves which pass through P_0:

C_u: $\vec{r}(u, v_0)$ hold v constant @ v_0

C_v: $\vec{r}(u_0, v)$ " u " @ u_0

Then, two tangent vectors at P_0 are

$$\vec{r}_u = \frac{\partial \vec{r}}{\partial u}(u_0, v_0) \ \& \ \vec{r}_v = \frac{\partial \vec{r}}{\partial v}(u_0, v_0)$$

If we wanted a normal vector to S @ P_0 we would take $\vec{r}_u \times \vec{r}_v$. As long as $\vec{r}_u \times \vec{r}_v \neq \vec{0}$, we're good.

<u>Def</u>: We call a parametrization $\vec{r}(u,v)$ of S <u>smooth</u> at P_0 if $\vec{r}_u \times \vec{r}_v \neq \vec{0}$.

So, a parametrization of the tangent plane is

$$T_{P_0}S(s,t) = \overrightarrow{OP_0} + s\vec{r}_u + t\vec{r}_v.$$